

CS3500
Computer Graphics
Module: Projective Geometry

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Projective Geometry of the Plane

- Points represented by: $[x \ y \ 1]^T$.
- Consider the line equation: $ax + by + c = 0$.
- Rewrite as: $[a \ b \ c] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \mathbf{L} \cdot \mathbf{P} = \mathbf{L}^T \mathbf{P} = 0$
- Lines are represented by 3-vectors, just like points. Scale is unimportant.
- Line equation: $\mathbf{L} \cdot \mathbf{P} = 0$. Describes all points incident on line \mathbf{L} or all lines passing through point \mathbf{P} !!

Points at Infinity

- $\mathbf{P} = [x \ y \ w]^T$ represents $(x/w, y/w)$.
What happens when $w \rightarrow 0$?
- Becomes **point at infinity** or **ideal** or **vanishing** point
- Points at infinity can be handled like any other in projective geometry.
- $[a \ b \ 0]^T$ are all points at infinity on the plane.
They together form a line at infinity.
- What is its representation? $[0 \ 0 \ 1]^T$.

View of Projective Representation

- $[x \ y \ w]$ represent rays from origin in a 3-space.
- Any cross section perpendicular to the w axis can describe the plane.
- Ideal points lie on the $w = 0$ plane.
- Lines are planes passing through the origin.

Line Joining 2 Points

- Let \mathbf{P}, \mathbf{Q} be two points. We have: $\mathbf{L} \cdot \mathbf{P} = \mathbf{L} \cdot \mathbf{Q} = 0$.
- Considering them as vectors in 3-space, we want to find a vector \mathbf{L} orthogonal to both \mathbf{P} and \mathbf{Q} .
- The cross-product $\mathbf{P} \times \mathbf{Q}$ is a solution. Thus, $\mathbf{L} = \mathbf{P} \times \mathbf{Q}$.
- Long route: $y = y_1 + (x - x_1) \frac{y_2 - y_1}{x_2 - x_1}$.
 $(y_2 - y_1)x - (x_2 - x_1)y + (x_2 - x_1)y_1 - (y_2 - y_1)x_1 = 0$
- $\mathbf{L} = [(y_2 - y_1) \quad -(x_2 - x_1) \quad x_2y_1 - x_1y_2]^T = \mathbf{P} \times \mathbf{Q}$.

- Line between $(5, 2)$ and $(3, 2)$: $\begin{bmatrix} i & j & k \\ 5 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix} = [0 \quad -2 \quad 4]^T$
 $\equiv [0 \quad 1 \quad -2]^T$. Same as $y = 2$.
- Ideal point of line $[0 \quad 1 \quad -2]^T$ is $[1 \quad 0 \quad 0]^T$.
- Line joining $[3 \quad 4 \quad 0]^T$ and $[2 \quad 3 \quad 0]^T$ is: $[0 \quad 0 \quad 1]^T$, the line at infinity.

Point of Intersection of 2 Lines

- Two lines L, M intersect in a point with $L \cdot P = M \cdot P = 0$.
- $P = L \times M$.
- $y = -(a_1x + c_1)/b_1$. And, $a_2x - b_2(a_1x + c_1)/b_1 + c_2 = 0$.
- $x = (b_2c_1 - b_1c_2)/(a_2b_1 - a_1b_2)$.
 $y = (a_1c_2 - a_2c_1)/(a_2b_1 - a_1b_2)$.
- $P = [(b_2c_1 - b_1c_2) \quad (a_1c_2 - a_2c_1) \quad (a_2b_1 - a_1b_2)]^T$

- Intersection of $x = 1$ and $y = 2$: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{bmatrix} = [1 \ 2 \ 1]^T$.
Same as $(1, 2)$.

- Intersection of $x = 1$ and $x = 2$: $\begin{bmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} = [0 \ 1 \ 0]^T$.
- Ideal point of line $[a \ b \ c]^T$ is $[b \ -a \ 0]^T$.

Conics: Second Order Entities

- General quadratic entity: $ax^2 + bxy + cy^2 + dx + ey + f = 0$.
- Rewrite using homogeneous coordinates as:
 $ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0$.
- Rewrite as: $[x \ y \ w] \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix} [x \ y \ w]^T = 0$
- A symmetric C represents a conic: $\mathbf{x}^T C \mathbf{x} = 0$.
Covers circle, ellipse, parabola, hyperbola, etc.

Properties of Conics

- Cx gives the tangent line to the conic at x .
- Dual conic: conic defined by its tangent lines!
- $l^T C^* l = 0$ where C^* is the adjoint matrix of C .
- If C is non-singular, $C^* \equiv C^{-1}$.
- Point of tangency of l and C is given $C^{-T} l = C^{-1} l$ due to symmetry.

Projective Transformations

- A general non-singular 3×3 matrix \mathbf{H} transforms points to other points. Overall scale of \mathbf{H} is unimportant.
- $\mathbf{x}' = \mathbf{H} \mathbf{x}$ gives the transformed point.
- $\mathbf{l}' = \mathbf{H}^{-\mathbf{T}} \mathbf{l}$ gives the transformed line.
- $\mathbf{C}' = \mathbf{H}^{-\mathbf{T}} \mathbf{C} \mathbf{H}^{-1}$ is the transformed conic.
- Linearity is preserved. \mathbf{P}' , \mathbf{Q}' , \mathbf{R}' collinear if \mathbf{P} , \mathbf{Q} , \mathbf{R} are.
- Such a transformation is called:
collineation, homography, projective transformation.

Isometric Transformation

- Transformations of the form, with $\delta = \pm 1$:

$$\begin{bmatrix} \delta \cos \theta & -\sin \theta & a \\ \delta \sin \theta & \cos \theta & b \\ 0 & 0 & 1 \end{bmatrix}$$

- Includes rotations, translations, reflections.
- Preserves distance measurements, angles, parallelism, etc.

Similarity Transformations

- Transformations of the form for nonzero s :

$$\begin{bmatrix} s\delta \cos \theta & -s \sin \theta & a \\ s\delta \sin \theta & s \cos \theta & b \\ 0 & 0 & 1 \end{bmatrix}$$

- Includes rotations, translations, uniform scaling
- Preserves angles, parallelism, ratio of distances.

Affine Transformations

- Transformations of the form:
$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$
- Includes rotations, translations, nonuniform scaling, shearing, etc.
- Preserves parallelism, ratio of lengths of parallel lines, ratio of areas, centroid.
- Points and lines at infinity map to other points and lines at infinity.

Projective Transformation

- Any general matrix \mathbf{H} , a general transformation.
- Finite points can map to ideal points and vice versa.
- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ maps every point to its ideal point!
- The impact of even slight projectivity is serious and very non-intuitive. Yet, that is what is done by a pin-hole camera.